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NORMAL ERGODIC ACTIONS AND EXTENSIONS

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ABSTRACT
We demonstrate that normal ergodic extensions of group actions are characterized as skew product actions given by cocycles into locally compact groups. As a consequence, Robert Zimmer's characterization of normal ergodic group actions is generalized to the noninvariant case. We also obtain the uniqueness theorem which generalizes the von Neumann Halmos uniqueness theorem and Zimmer's uniqueness theorem for normal actions with relative discrete spectrum.

Rober Zimmer introduced the notions of normal ergodic group actions and extensions in [18]. He later characterized normal ergodic group actions when the measure is σ-finite and invariant. See [20]. He also conjectured that normal ergodic extensions of ergodic group actions could be characterized as skew product actions constructed from cocycles into locally compact groups. Our intent is to obtain this result. As a consequence we generalize Zimmer's characterization of normal ergodic actions to the noninvariant case. We also obtain the uniqueness theorem; that is we show two normal ergodic extensions of the same group action which induce equivalent unitary Hilbert bundle representations are conjugate. This is the generalization of the von Neumann Halmos uniqueness theorem for ergodic transformations with pure point spectrum and Zimmer's uniqueness theorem for normal actions having relative discrete spectrum.

Let X be a standard Borel space and let μ be a σ-finite measure on X. Let G be a second countable locally compact group. By a Borel action of G on X we mean a Borel mapping $x, g \mapsto x \cdot g$ from $X \times G$ to X such that $x \cdot e = x$ and $(x \cdot g_1) \cdot g_2 = x \cdot g_1 g_2$ for all x, g_1 and g_2. The measure μ is assumed to be quasi-invariant; that is $\mu \cdot g(E) \equiv \mu(E \cdot g^{-1}) = 0$ iff $\mu(E) = 0$. The action of G on X then induces a Boolean action of G on the measure algebra $M(X, \mu)$, and

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this action essentially determines the point action on X. When we speak of two G spaces being conjugate, we shall mean isomorphism of the actions induced on their measure algebras. We recall that the action of G on (X, μ) is ergodic if there are no proper invariant elements in the measure algebra.

Suppose (X, μ) and (Y, ν) are G spaces. A G equivariant map $p: X \to Y$ with $p_*\mu = \mu \circ p^{-1} \sim \nu$ is called an extension of Y to X. The extension p of Y to X induces a unitary Hilbert bundle representation of $Y \times G$. Namely, let $\mu = \int \mu_y d\nu(y)$ be the disintegration of the measure μ over the fibers of p. This yields a direct integral decomposition $\int^{\oplus} L^2(\mu_y) d\nu(y)$ of $L^2(\mu)$. Since $\mu \cdot g \sim \mu$, $\mu_y \cdot g \sim \mu_{y \cdot g}$ a.e. y for each g. Define $R(y, g): L^2(\mu_{y \cdot g}) \to L^2(\mu_y)$ by

$$R(y, g) f(x) = \frac{d\mu_{y \cdot g}}{d\mu_y \cdot g} (xg)^{1/2} f(x \cdot g).$$

Then $R(y, g)$ is unitary a.e. y for each g and satisfies $R(y, g_1) R(y \cdot g_1, g_2) = R(y, g_1 g_2)$ a.e. y for all g_1 and g_2. R is called the induced Hilbert bundle representation of $Y \times G$. The extension X of Y given by p is called normal if there exists a Borel field $x \mapsto U(x)$ of unitaries with $U(x)$ mapping $L^2(\mu_{p(x)})$ to a fixed Hilbert space H such that $U(x) R(p(x), g) U(x \cdot g)^{-1} = I$ a.e. x for each g.

An action of G on X is normal provided it is a normal extension of the trivial action. That is there is a Borel field $x \mapsto U(x)$ of unitaries from $L^2(\mu)$ to a fixed Hilbert space H such that $U(x) R(g) U(x \cdot g)^{-1} = I$ a.e. x for each g, where here

$$R(g) f(x) = \frac{d\mu}{d\mu \cdot g} (x \cdot g)^{1/2} f(x \cdot g).$$

Suppose now $p': (X', \mu') \to Y$ is another extension of Y. It induces a unitary Hilbert bundle representation R' of $Y \times G$ on the bundle $y \mapsto L^2(\mu'_y)$. The representations R and R' are said to be equivalent if there exists a Borel field $y \mapsto U(y)$ of unitaries from $L^2(\mu_y)$ to $L^2(\mu'_y)$ such that $U(y) R(y, g) = R'(y, g) U(y \cdot g)$ a.e. y for each g.

We recall a cocycle from $Y \times G$ into a locally compact group is a Borel function a from $Y \times G$ into the group satisfying $a(y, g_1) a(y \cdot g_1, g_2) = a(y, g_1 g_2)$ a.e. y for each g_1 and g_2.

We can now state our results.

THEOREM 1. *Let $p: X \to Y$ be an ergodic normal extension. Then there exists a second countable locally compact group H and a cocycle a from $Y \times G$ into H such that the action of G on X is conjugate to the action of G on $(H \times Y$, Haar measure $\times \nu)$ defined by $(h, y) \cdot g = (ha(y, g), y \cdot g)$. Furthermore, under the conjugacy, p is carried to the projection $(h, y) \mapsto y$.*

THEOREM 2 (Uniqueness Theorem). *Suppose $p: X \to Y$ and $p': X' \to Y$ are normal ergodic extensions and induce equivalent unitary Hilbert bundle representations of $Y \times G$. Then the G actions on X and X' are conjugate.*

COROLLARY 1. *Suppose the action of G on (X, μ) is normal and ergodic. Then there is a locally compact group H and a continuous homomorphism a from G into H with dense range such that the action of G on X is conjugate to the action of G on $(H$, Haar measure$)$ defined by $h \cdot g = ha(g)$.*

COROLLARY 2. *Suppose X and X' are normal ergodic G spaces which induce equivalent unitary representations of G. Then they are conjugate.*

Demonstration of Theorem 1

Before beginning the proof of Theorem 1 we give a preliminary result. Let Y be a standard Borel space with finite measure ν. Let $\mathcal{I}(Y, \nu)$ be the set of all Borel isomorphisms φ of Y such that $\varphi_* \nu \sim \nu$, where we identify any two equal a.e. Give $\mathcal{I}(Y, \nu)$ the smallest Borel structure for which $\varphi \mapsto \int f \circ \varphi(y) h(y) d\nu(y)$ is Borel for all real valued bounded Borel functions f and h on Y. Then $\mathcal{I}(Y, \nu)$ is Borel isomorphic to the strongly closed subgroup of the unitary group of $L^2(Y, \nu)$ consisting of those unitaries U such that $U1 \geqq 0$ and $UL^\infty(Y, \nu)U^{-1} = L^\infty(Y, \nu)$. The isomorphism is given by the map $\varphi \to L_\varphi$ where

$$L_\varphi f = \left(\frac{d\mu \circ \varphi^{-1}}{d\nu}\right)^{1/2} f \circ \varphi^{-1}.$$

PROPOSITION 1. *Let $p: X \to Y$ be an extension of ergodic G space Y. Then there exist a standard Borel space S, a finite measure m on S, a Borel isomorphism $\varphi: S \times Y \to X$, and a Borel cocycle $a: Y \times G \to \mathcal{I}(S, m)$ such that*

(a) $\varphi_* m \times \nu \sim \mu$,

(b) $p \circ \varphi(s, y) = y$ *a.e.* s, y,

(c) $\varphi(a(y, g)^{-1}(s), y \cdot g) = \varphi(s, y) \cdot g$ *a.e.* s, y *for each* g.

PROOF. Sketch. Let $\mu = \int \mu_y d\nu(y)$ be the disintegration of μ over the fibers of p. Each $p^{-1}(y)$ is a standard Borel space with measure μ_y. For each n, an integer, let J_n be a standard measure space with $|n|$ atoms and continuous part only if $n \geqq 0$. Then $Y_n = \{y : (p^{-1}(y), \mu_y)$ is essentially isomorphic to $J_n\}$ is a G invariant Borel set. Since the action of G on Y is ergodic, Y_n is conull for some n. Using the von Neumann selection, theorem, there is a Borel field $y \mapsto \varphi_y$ of functions from $p^{-1}(y)$ to J_n such that φ_y is an isomorphism and carries the measure for J_n to a measure equivalent to the measure μ_y for a.e. y in Y_n. By "integrating" the φ_y's, one obtains a Borel function $\varphi : J_n \times Y \to X$ which is essentially an isomorphism and carries the product measure on $J_n \times Y$ to a measure equivalent to μ. Let $S = J_n$ and m be the measure of J_n. The rest follows easily. Q.E.D.

Suppose now (X, μ) is a normal ergodic extension of Y. By Proposition 1, we may assume $X = S \times Y$, $\mu = m \times \nu$, and $(s, y) \cdot g = (b(y, g)^{-1}(s), y \cdot g)$ a.e. s, y for each g where $b : Y \times G \to \mathcal{I}(S, m)$ is a Borel cocycle. One then can note that the induced unitary Hilbert bundle representation of $Y \times G$ is then actually a unitary representation of $Y \times G$ on $L^2(S, m)$. Namely, $R(y, g) = L_{b(y,g)}$. Since we are assuming the extension is normal, one has a Borel function $(s, y) \mapsto U(s, y)$ where each $U(s, y)$ is a unitary operator from $L^2(S, m)$ onto a fixed Hilbert space $\mathbf{H}$ that satisfies $U(s, y)R(y, g)U(b(y, g)^{-1}s, y \cdot g)^{-1} = I$ a.e. s, y for each g.

Effros in [1] showed there exists a standard Borel structure on the space $\mathcal{V}$ of von Neumann algebras on $\mathbf{H}$ such that $s, y \mapsto \mathcal{M}(s, y) = U(s, y)L^\infty(S, m)U(s, y)^{-1}$ is Borel. Since $R(y, g) = L_{b(y,g)} \in \mathcal{I}(S, m)$, it follows that $\mathcal{M}((s, y) \cdot g) = \mathcal{M}(s, y)$ a.e. s, y for each g. Since the G action on $S \times Y$ is ergodic, $\mathcal{M}(s, y)$ is constant a.e. s, y. Hence there exists a unitary V from $\mathbf{H}$ to $L^2(S, m)$ such that $VU(s, y)L^\infty(S, m)U(s, y)^{-1}V^{-1} = L^\infty(S, m)$ a.e. s, y. By redefining $U(s, y)$ to be $VU(s, y)$ and redefining on a set of measure 0, we may assume $U(s, y) : L^2(S, m) \to L^2(S, m)$ and $U(s, y)L^\infty(S, m)U(s, y)^{-1} = L^\infty(S, m)$. Now for each s, y, there exists an essentially unique Borel complex valued function $u_{s,y}$ on S with $|u_{s,y}| \equiv 1$ such that $u_{s,y} \cdot U(s, y)1 \geqq 0$. Let $M_{u_{s,y}}f = u_{s,y} \cdot f$ for $f \in L^2(S, m)$. Then $M_{u_{s,y}}U(s, y) \in \mathcal{I}(S, m)$ for every s, y. Hence there exists a $\varphi_{s,y} \in \mathcal{I}(S, m)$ such that $U(s, y) = M_{\bar{u}_{s,y}}L_{\varphi_{s,y}}$. Both $s, y \mapsto M_{\bar{u}_{s,y}}$ and $s, y \mapsto L_{\varphi_{s,y}}$ are strongly Borel. Since $U((s, y) \cdot g) = U(s, y)L_{b(y,g)}$, we see $M_{\bar{u}_{(s,y)\cdot g}} = M_{\bar{u}_{s,y}}$ and $L_{\varphi(s,y)\cdot g} = L_{\varphi_{s,y}}L_{b(y,g)}$ a.e. s, y for each g. We hence redefine $U(s, y)$ to be $L_{\varphi_{s,y}}$. We have $\varphi_{(s,y)\cdot g} = \varphi_{s,y}b(y, g)$ a.e. s, y for each g. Since $s, y \mapsto \varphi_{s,y}$ and $s, y \mapsto \varphi_{s,y}^{-1}$ are Borel mappings of $S \times Y$ into $\mathcal{I}(S, m)$, there exist Borel functions φ and φ^{-1} from $S \times Y \times S$ into S such that $\varphi(s, y, t) = \varphi_{s,y}(t)$ and $\varphi^{-1}(s, y, t) = \varphi_{s,y}^{-1}(t)$ a.e. t a.e. s, y. Furthermore, φ and φ^{-1} *satisfy* $\varphi(b(y, g)^{-1}s, y \cdot g, t) = \varphi(s, y, b(y, g)t)$ and $\varphi^{-1}(b(y, g)^{-1}s, y \cdot g, t) = b(y, g)^{-1}\varphi^{-1}(s, y, t)$ a.e. t a.e. s, y for each g.

Proposition 2. *$U(s, y)U(r, y)^{-1} = U(\varphi_{r,z}^{-1} \circ \varphi(r, y, s), z)U(r, z)^{-1}$ a.e. s, y, r, z.*

Proof. Define $F(x, t) = U(\varphi^{-1}(x, t), \pi_2 x)U(x)^{-1}$ for $x \in S \times Y$ and $t \in S$. Then F is Borel and

$$\begin{aligned} F(x\cdot g,t) &= U(\varphi^{-1}(x\cdot g,t),\pi_2 x\cdot g)U(x\cdot g)^{-1} \\ &= U(b(\pi_2 x,g)^{-1}\varphi^{-1}(x,t),\pi_2 x\cdot g)U(x\cdot g)^{-1} \\ &= U(\varphi^{-1}(x,t),\pi_2 x)L_{b(\pi_2 x,g)}L_{b^{-1}(\pi_2 x,g)}U(x)^{-1} \\ &= F(x,t) \qquad \text{a.e. } x \text{ a.e. } t \text{ for each } g. \end{aligned}$$

Since G is locally compact, it follows $x \to F(x,t)$ is G invariant for a.e. t. Hence there exists a Borel function $A : T \to \mathcal{I}(S,m)$ satisfying $A(t) = F(x,t)$ a.e. x, t.

Choose $r, z \in S \times Y$ so that $\varphi_{r,y} = \varphi(r,y,\cdot)$, $\varphi_{r,y}^{-1} = \varphi^{-1}(r,y,\cdot)$ a.e. y, $\varphi_{r,z} = \varphi(r,z,\cdot)$, $\varphi_{r,z}^{-1} = \varphi^{-1}(r,z,\cdot)$, $U(\varphi^{-1}(r,y,t),y)U(r,y)^{-1} = A(t)$ a.e. t, y and $U(\varphi^{-1}(r,z,t),z)U(r,z)^{-1} = A(t)$ a.e. t. Hence $U(t,z)U(r,z)^{-1} = A(\varphi(r,z,t)) = U(\varphi^{-1}(r,y,\varphi(r,z,t)),y)U(r,y)^{-1}$ a.e. t, y. The results follows. Q.E.D.

Choose $r, z \in S \times Y$ such that:

(a) $\varphi_{r,y} = \varphi(r,y,\cdot)$ a.e. y,

(b) $\varphi_{r,y}^{-1} = \varphi^{-1}(r,y,\cdot)$ a.e. y,

(c) $U(t,y)U(r,y)^{-1} = U(\varphi_{r,z}^{-1}\circ\varphi(r,y,t),z)U(r,z)^{-1}$ a.e. t, y.

Define $\bar{U}(t,y) = U(r,z)^{-1}U(t,y)U(r,y)^{-1}U(r,z)$. Note $\bar{U}(r,y) = I$ for all y.

LEMMA 3. *$\bar{U}(t,y) = \bar{U}(\varphi_{r,z}^{-1}\circ\varphi(r,y,t),z)$ a.e. t, y.*

PROOF.

$$\begin{aligned} \bar{U}(t,y) &= U(r,z)^{-1}U(t,y)U(r,y)^{-1}U(r,z) \\ &= U(r,z)^{-1}U(\varphi_{r,z}^{-1}\circ\varphi(r,y,t),z)U(r,z)^{-1}U(r,z) \\ &= \bar{U}(\varphi_{r,z}^{-1}\circ\varphi(r,y,t),z) \qquad \text{a.e. } t, y. \end{aligned}$$

LEMMA 4. *$\bar{U}((t,y)\cdot g) = \bar{U}(t,y)L_{a(y,g)}$ a.e. t, y for each g, where $a(y,g) = \varphi_{r,z}^{-1}\circ\varphi_{r,y}\circ b(y,g)\circ(\varphi_{r,z}^{-1}\circ\varphi_{r,y\cdot g})^{-1}$.*

PROOF.

$$\begin{aligned} &\bar{U}((t,y)\cdot g) \\ &= U(r,z)^{-1}U((t,y)\cdot g)U(r,y\cdot g)^{-1}U(r,z) \\ &= U(r,z)^{-1}U(t,y)U(r,y)^{-1}U(r,z)U(r,z)^{-1}U(r,y)L_{b(y,g)}U(r,y\cdot g)^{-1}U(r,z) \\ &= \bar{U}(t,y)L_{a(y,g)} \qquad \text{a.e. } t, y. \end{aligned}$$

Define $\psi \in \mathcal{I}(S \times Y, m \times \nu)$ by $\psi(s,y) = (\varphi_{r,z}^{-1}\circ\varphi(r,y,s),y)$. Clearly $\psi^{-1}(s,y) = (\varphi^{-1}(r,y,\varphi_{r,z}(s)),y)$.

LEMMA 5. *$\psi(\psi^{-1}(s,y)\cdot g) = (a(y,g)^{-1}s, y\cdot g)$ a.e. s, y for each g.*

PROOF.

$$\psi(b(y,g)^{-1}s, y\cdot g) = (\varphi_{r,z}^{-1}\circ\varphi(r, y\cdot g, b(y,g)^{-1}s), y\cdot g)$$
$$= (\varphi_{r,z}^{-1}\circ\varphi_{r,y\cdot g}\circ b(y,g)^{-1}\circ\varphi_{r,y}^{-1}\circ\varphi_{r,z}\circ\varphi_{r,z}^{-1}\circ\varphi_{r,y}(s), y\cdot g)$$
$$= (a(y,g)^{-1}\varphi_{r,z}^{-1}\circ\varphi(r,y,s), y\cdot g). \qquad \text{Q.E.D.}$$

Hence we see (s, y), $g \mapsto (a(y,g)^{-1}s, y\cdot g)$ defines a Borel almost action of G on $S\times Y$ conjugate to the action of G on X.

Define V on $S\times Y$ by $V=\bar{U}\circ\psi^{-1}$.

LEMMA 6. *$V(a(y,g)^{-1}s, y\cdot g) = V(s,y)L_{a(y,g)}$ a.e. s, y for each g.*

PROOF.

$$V(a(y,g)^{-1}s, y\cdot g) = \bar{U}(\psi^{-1}(a(y,g)^{-1}s, y\cdot g))$$
$$= \bar{U}(\psi^{-1}(s,y)\cdot g)$$
$$= \bar{U}(\psi^{-1}(s,y))L_{a(y,g)} \quad \text{a.e. } s, y. \qquad \text{Q.E.D.}$$

PROPOSITION 7. *$V(s,y) = V(s,z)$ a.e. s, y.*

PROOF.

$$V(s,y) = \bar{U}\circ\psi^{-1}(s,y)$$
$$= \bar{U}(\varphi^{-1}(r,y,\varphi_{r,z}(s)), y)$$
$$= \bar{U}(\varphi_{r,z}^{-1}\circ\varphi(r,y,\varphi^{-1}(r,y,\varphi_{r,z}(s))), z)$$
$$= \bar{U}(\varphi_{r,z}^{-1}\circ\varphi_{r,z}(s), z) = \bar{U}(s,z)$$
$$= V(s,z)$$

a.e. s, y where the equalities follow by Lemma 3 and (a) and (b).

We hence have the following situation: the action of G on X is conjugate to the action of G on $S\times Y$ defined by $(s,y)\cdot g = (a(y,g)^{-1}s, y\cdot g)$ where $a: Y\times G \to \mathcal{I}(S,m)$ is Borel; p is carried by the conjugacy to π_2; and there exists a Borel map $U: S\to\mathcal{I}(S,m)$ such that $U(a(y,g)^{-1}s) = U(s)L_{a(y,g)}$ a.e. s, y for each g.

Let $U(s) = L_{\varphi_s}$ where $\varphi_s \in \mathcal{I}(S,m)$. There exist Borel functions φ and φ^{-1} mapping $S\times S$ to S such that $\varphi(s,t) = \varphi_s(t)$, $\varphi^{-1}(s,t) = \varphi_s^{-1}(t)$ a.e. t a.e. s. Since $U(a(y,g)^{-1}s) = U(s)L_{a(y,g)}$ a.e. s, y, $\varphi(s, a(y,g)t) = \varphi(a(y,g)^{-1}s, t)$ and $a(y,g)^{-1}\varphi^{-1}(s,t) = \varphi^{-1}(a(y,g)^{-1}s, t)$ a.e. s, y, t.

LEMMA 8. *There exists a Borel function $A: S\to\mathcal{I}(S,m)$ such that $U(\varphi^{-1}(s,t)) = A(t)U(t)U(s)$ a.e. s, t.*

PROOF. Define $F(t,s,y)=U(\varphi^{-1}(s,t))U(s)^{-1}U(t)^{-1}$.

$$\begin{aligned} F(t,a(y,g)^{-1}s,y\cdot g) &= U(\varphi^{-1}(a(y,g)^{-1}s,t))U(a(y,g)^{-1}s)^{-1}U(t)^{-1} \\ &= U(a(y,g)^{-1}\varphi^{-1}(s,t))U(a(y,g)^{-1}s)^{-1}U(t)^{-1} \\ &= U(\varphi^{-1}(s,t))L_{a(y,g)}L^{-1}_{a(y,g)}U(s)^{-1}U(t)^{-1} \\ &= F(t,s,y) \qquad \text{a.e. } t,s,y. \end{aligned}$$

The result follows by ergodicity. Q.E.D.

COROLLARY 9. *$U(\varphi(s,t))=A(\varphi(s,t))^{-1}U(t)U(s)^{-1}$ a.e. s,t.*

PROPOSITION 10. *U may be modified so that $U(\varphi^{-1}(s,t))=U(t)U(s)$ and $U(\varphi(s,t))=U(t)U(s)^{-1}$ a.e. s,t.*

PROOF. Let S_0 be a conull Borel subset of S such that $\varphi_s=\varphi(s,\cdot)$, $\varphi_s^{-1}=\varphi^{-1}(s,\cdot)$, $U(\varphi^{-1}(s,t))=A(t)U(t)U(s)$, and $U(\varphi(s,t))=A(\varphi(s,t))^{-1}U(t)U(s)^{-1}$ a.e. t if $s\in S_0$.

Choose $r\in S_0$. Define $\bar U(s)=U(r)^{-1}U(s)$. Hence $\bar\varphi_s=\varphi_r^{-1}\circ\varphi_s$. Take $\bar\varphi(s,t)=\varphi_r^{-1}\circ\varphi(s,t)$ and $\bar\varphi^{-1}(s,t)=\varphi^{-1}(s,\varphi_r(t))$.

Let $s\in S_0$.

$$\begin{aligned} \bar U(\bar\varphi^{-1}(s,t)) &= U(r)^{-1}U(\varphi^{-1}(s,\varphi_r(t))) \\ &= U(r)^{-1}A(\varphi(r,t))U(\varphi(r,t))U(s) \\ &= U(r)^{-1}A(\varphi(r,t))A(\varphi(r,t))^{-1}U(t)U(r)^{-1}U(s) \\ &= \bar U(t)\bar U(s) \qquad \text{a.e. } t. \end{aligned}$$

Hence the first statement holds; the second is verified similarly.

We define Borel almost G actions $\cdot$ and $*$ on $\tilde X=S\times S\times Y$ by $(s,t,y)\cdot g=(a(y,g)^{-1}s,a(y,g)^{-1}t,y\cdot g)$ and $(y,t,y)*g=(a(y,g)^{-1}s,t,y)$.

PROPOSITION 11. *$s\mapsto\varphi^{-1}(s,t)$ is measure class preserving m a.e. t.*

PROOF. Define G and H in $\mathcal{I}(\tilde X,m\times m\times\nu)$ by $G(s,t,y)=(s,\varphi(s,t),y)$ and $H(s,t,y)=(s,\varphi^{-1}(s,t),y)$. Clearly $H^{-1}=G$. Also

$$\begin{aligned} G((s,t,y)\cdot g) &= G(a(y,g)^{-1}s,a(y,g)^{-1}t,y\cdot g) \\ &= (a(y,g)^{-1}s,\varphi(a(y,g)^{-1}s,a(y,g)^{-1}t),y\cdot g) \\ &= (a(y,g)^{-1}s,\varphi(s,t),y\cdot g) \\ &= G(x,t,y)*g \qquad \text{a.e. } s,t,y \text{ for each } g. \end{aligned}$$

$\pi_2 : \tilde{X} \to S$ gives an ergodic deomposition of the action $*\cdot$. Hence $\pi_2 \circ G$ gives an ergodic decomposition of the action on $\tilde{X}$. The map A defined by $A(s,t,y) = (t,s,y)$ preserves the action $\cdot$; hence $\pi_2 \circ G \circ A$ gives another ergodic decomposition of the action $\cdot$. By uniqueness of ergodic decompositions (see [12]) there exists a ψ in $\mathcal{I}(S,m)$ such that $\psi \circ \pi_2 \circ G = \pi_2 \circ G \circ A$. Let $m \times m \times \nu$ be disintegrated over $\pi_2 \circ G$; namely $m \times m \times \nu = \int \mu_t dm(t)$ where each μ_t is concentrated on $(\pi_2 \circ G)^{-1}(t)$ m a.e. t. Then $m \times m \times \nu = \int \mu_{\psi^{-1}(t)} d\psi_* m(t)$ and $\mu_{\psi^{-1}(t)}$ is concentrated on $(\pi_2 \circ G)^{-1}(\psi^{-1}(t)) = (\psi \circ \pi_2 \circ G)^{-1}(t)$ a.e. t. But $m \times m \times \nu = A_*(m \times m \times \nu) = \int A_* \mu_t dm(t)$ and $A_*\mu_t$ is concentrated on $A(\pi_2 \circ G)^{-1}(t) = (\pi_2 \circ G \circ A)^{-1}(t) = (\psi \circ \pi_2 \circ G)^{-1}(t)$ a.e. t. Hence $A_*\mu_t \sim \mu_{\psi^{-1}(t)}$ a.e. t. Since $A = A^{-1}$, $A_*\mu_t \sim \mu_{\psi(t)}$ a.e. t.

Now $m \times m \times \nu = \int m \times \varepsilon_t \times \nu dm(t)$. Hence $m \times m \times \nu \sim \int H_*(m \times \varepsilon_t \times \nu) dm(t)$ and $H_*(m \times \varepsilon_t \times \nu)$ is concentrated on $H\pi_2^{-1}(t) = (\pi_2 \circ G)^{-1}(t)$ a.e. t. Hence $H_*(m \times \varepsilon_t \times \nu) \sim \mu_t$ a.e. t. Hence $A_* H_*(m \times \varepsilon_t \times \nu) \sim H_*(m \times \varepsilon_{\psi(t)} \times \nu)$ a.e. t.

Suppose $A_* H_*(m \times \varepsilon_t \times \nu) \sim H_*(m \times \varepsilon_{\psi(t)} \times \nu)$. Then

$$\begin{aligned} m\{s : \varphi^{-1}(s,t) \in E\} > 0 &\text{ iff } (m \times \varepsilon_t \times \nu)(H^{-1}(S \times E \times Y)) > 0 \\ &\text{ iff } (m \times \varepsilon_t \times \nu)(H^{-1}A^{-1}(E \times S \times Y)) > 0 \\ &\text{ iff } (m \times \varepsilon_{\psi(t)} \times \nu)(H^{-1}(E \times S \times Y)) > 0 \\ &\text{ iff } m(E) > 0. \end{aligned}$$

Q.E.D.

Define $\tilde{\varphi} : S \to \mathcal{I}(S,m)$ by $\tilde{\varphi}(s) = \varphi_s$. Let $m^* = \tilde{\varphi}_* m$.

PROPOSITION 12. *For m a.e. t, the measure class of m^* is invariant under the mappings $\psi \mapsto \psi \circ \varphi_t$ and $\psi \mapsto \varphi_t \circ \psi$.*

PROOF. Since $U(t)U(s) = U(\varphi^{-1}(s,t))$, $\varphi_t \circ \varphi_s = \varphi_{\varphi^{-1}(s,t)}$ a.e. s,t. Choose t such that $\varphi_t \circ \varphi_s = \varphi_{\varphi^{-1}(s,t)}$, $\varphi_s \circ \varphi_t = \varphi_{\varphi^{-1}(t,s)}$, $\varphi^{-1}(t,s) = \varphi_t^{-1}(s)$ a.e. s, and $s \mapsto \varphi^{-1}(s,t)$ preserves the measure class of m. Then $m^*(N) > 0$ iff $m\{s : \varphi_s \in N\} > 0$ iff $m\{s : \varphi_{\varphi^{-1}(s,t)} \in N\} > 0$ iff $m\{s : \varphi_t \circ \varphi_s \in N\} > 0$ iff $m^*\{\psi : \varphi_t \circ \psi \in N\} > 0$. Also $m^*(N) > 0$ iff $m\{s : \varphi_{\varphi^{-1}(t,s)} \in N\} > 0$ iff $m\{s : \varphi_s \circ \varphi_t \in N\} > 0$ iff $m^*\{\psi : \psi \circ \varphi_t \in N\} > 0$. Q.E.D.

THEOREM 13. *Let $H = \{\psi \in \mathcal{I}(S,m) : \tau \mapsto \tau \circ \psi$ and $\tau \mapsto \psi \circ \tau$ preserve the measure class of m^*. Then H has a second countable locally compact topology compatible with the group structure such that m^* is equivalent to Haar measure on H.*

PROOF. Let $\mathcal{M}$ be the space of probability measures on $\mathcal{I}(S, m)$ equipped with smallest Borel structure such that $\mu \mapsto \mu(E)$ is Borel for each Borel set E in $\mathcal{I}(S, m)$. Then $\mathcal{M}$ is standard Borel space and the actions of $\mathcal{I}(S, m)$ defined on $\mathcal{M}$ by $\mu \cdot \varphi(E) = \mu(E\varphi^{-1})$ and $\varphi \cdot \mu(E) = \mu(\varphi^{-1}E)$ are Borel. Furthermore, $\{\mu \in \mathcal{M} : \mu \sim m^*\}$ is Borel by lemma 1.1 of [11]. Hence H is a Borel subgroup of the Polish group $\mathcal{I}(S, m)$. By Proposition 12, m^* is concentrated on H. By lemma 8.33 of [15], m^* is equivalent to a right invariant σ-finite measure on H. The result follows by theorem 8.33 of [15]. Q.E.D.

PROPOSITION 14. *$a(y, g) \in H$ ν a.e. y for each g.*

PROOF. $\psi \rightarrow \psi \circ a(y, g)$ preserves the measure class of m^* ν a.e. y, for $\varphi_s \circ a(y, g) = \varphi_{a(y,g)^{-1}s}$ a.e. s, y and $a(y, g) \in \mathcal{I}(S, m)$. Furthermore, $\psi \mapsto a(y, g) \circ \psi$ is a composition of the maps $\psi \mapsto \psi^{-1}$, $\psi \mapsto \psi \circ a(y, g)^{-1} = \psi \circ a(yg, g^{-1})$, and $\psi \mapsto \psi^{-1}$ of H, and hence preserves the measure class of m^* ν a.e. y. Q.E.D.

PROPOSITION 15. *If $\psi \in H$, $\varphi_s \circ \psi = \varphi_{\psi^{-1}(s)}$ m a.e. s.*

PROOF. $\{\psi : \varphi_s \circ \psi = \varphi_{\psi^{-1}(s)}$ a.e. $s\}$ is a Borel subgroup of H which is conull since $\varphi_s \circ \varphi_t = \varphi_{\varphi^{-1}(t,s)} = \varphi_{\varphi_t^{-1}(s)}$ a.e. s, t. Q.E.D.

Define a Boolean action of H on $M(S, m)$ by $E \cdot \psi = \psi^{-1}(E)$.

PROPOSITION 16. *The Boolean action of H on $M(S, m)$ is ergodic.*

PROOF. Suppose $E \in M(S, m)$ and $E \cdot \psi = E$ for all $\psi \in H$. Let $W = E \times Y \in M(S \times Y, m \times \nu)$. Since $a(y, g) \in H$ a.e. y for each g, W is invariant under the ergodic G action $(s, y)g = (a(y, g)^{-1}s, y \cdot g)$. Hence $E = 0$ or $E = S$. Q.E.D.

There is no loss in assuming $a(y, g)$ is in H for all y, g for one can redefine on a set of measure 0 and note that $\{g : (s, y) \cdot g = (a(y, g)^{-1}s, y \cdot g)$ a.e. $s, y\}$ is a conull multiplicatively closed subset of G. We may also assume φ_s is in H for all s by redefining on a set of measure 0.

We now conclude the proof of Theorem 1. $\tilde{\varphi}$ is a Borel map from S into H such that $\tilde{\varphi}_* m = m^*$ and $\varphi_s \circ \psi^{-1} = \varphi_{\psi(s)}$ for any ψ in H. Hence $\tilde{\varphi}$ is H equivariant from the Boolean action of H on S to the action of right translation of H on H. In fact,

$$\tilde{\varphi}^{-1}(E\psi) = \{s : \varphi_s \circ \psi^{-1} = \varphi_{\psi(s)} \in E\} = \psi^{-1}\tilde{\varphi}^{-1}(E) = \tilde{\varphi}^{-1}(E) \cdot \psi.$$

But any H equivariant map into the H space H is essentially one-to-one.

Define $J : S \times Y \rightarrow H \times Y$ by $J(s, y) = (\tilde{\varphi}(s), y)$. Then $J_*(m \times \nu) = m^* \times \nu$, J

is essentially one-to-one, and $J((s,y)\cdot g)=J(a(y,g)^{-1}s, y\cdot g)=(\varphi_{a(y,g)^{-1}s}, y\cdot g)=(\varphi_s\circ a(y,g), y\cdot g)=(\tilde{\varphi}_s a(y,g), y\cdot g)$ a.e. s,y for each g. This concludes the proof.

The uniqueness theorem

Theorem 2 follows from Theorem 1 and the following proposition. Let H and K be second countable locally compact groups; let (Y,ν) be a G space; and let $a: Y\times G\to H$ and $b: Y\times G\to K$ be cocycles. Suppose the almost actions of G on $H\times Y$ and $K\times Y$ defined by $(h,y)\cdot g=(ha(y,t), y\cdot g)$ and $(k,y)\cdot g=(kb(y,g), y\cdot g)$ are ergodic.

PROPOSITION. *If the unitary representations of $Y\times G$ induced by the extensions $(h,y)\mapsto y$ and $(k,y)\mapsto y$ are equivalent, the G spaces $H\times Y$ and $K\times Y$ are conjugate.*

PROOF. By $\mathcal{I}(H)$ we shall mean $\mathcal{I}(H$, Haar measure). The representation of $Y\times G$ induced by $(h,y)\mapsto y$ is $(y,g)\mapsto R_{a(y,g)}$ where $R_{a(y,g)}$ is the unitary operator on $L^2(H)$ defined by $R_{a(y,g)}f(h)=f(ha(y,g))$. Hence $R_{a(y,g)}=L_{r_{a(y,g)^{-1}}}$ where $r_h(h')=h'h$. Similarly $(y,g)\mapsto R_{b(y,g)}=L_{r_{b(y,g)^{-1}}}$ is the unitary representation of $Y\times G$ induced by extension $(k,y)\mapsto y$.

Since the induced representations are equivalent, there exists a Borel map $y\mapsto W(y)$ where $W(y)$ is a unitary from $L^2(H)$ to $L^2(K)$ which satisfies $W(y)R_{a(y,g)}=R_{b(y,g)}W(y\cdot g)$ a.e. y for each g.

Define $U(k,y)=R_kW(y)$. Set $\mathcal{M}(k,y)=U(k,y)L^\infty(H)U(k,y)^{-1}$. $\mathcal{M}$ is G invariant. By ergodicity, there exists a unitary V from $L^2(K)$ to $L^2(H)$ such that $VU(k,y)L^\infty(H)U(k,y)^{-1}V^{-1}=L^\infty(H)$ a.e. k,y. Hence for a.e. k,y there exists a $u_{k,y}$ in $L^\infty(H)$ with $|u_{k,y}|\equiv 1$ and a $\varphi_{k,y}$ in $\mathcal{I}(H)$ such that $VU(k,y)=M_{u_{k,y}}L_{\varphi_{k,y}}$. But $VU((k,y)\cdot g)=VR_kR_{b(y,g)}W(y\cdot g)=VR_kW(y)R_{a(y,g)}=U(k,y)R_{a(y,g)}$ a.e. y,g. Hence $M_{u_{(k,y)\cdot g}}=M_{u_{(k,y)}}$ and $\varphi_{(k,y)\cdot g}=\varphi_{k,y}\circ r_{a(y,g)^{-1}}$ a.e. k,y for each g. Therefore we may assume $VU(k,y)=L_{\varphi_{k,y}}$ for all k and y and $\varphi_{(k,y)\cdot g}=\varphi_{k,y}\circ r_{a(y,g)^{-1}}$ a.e. k,y for each g. Hence $\varphi^{-1}_{k,y}(h)a(y,g)=\varphi^{-1}_{(k,y)\cdot g}(h)$ a.e. h a.e. k,y for each g.

Choose h_0 such that $\varphi^{-1}_{k,y}(h_0)a(y,g)=\varphi^{-1}_{(k,y)\cdot g}(h_0)$ a.e. k,y,g. Define $F(k,y)=r_{\varphi^{-1}_{k,y}(h_0)^{-1}}\circ\varphi^{-1}_{k,y}$. Then

$$F(k,y)\cdot g=r_{a(y,g)^{-1}\varphi^{-1}_{k,y}(h_0)^{-1}}\circ\varphi^{-1}_{(k,y)\cdot g}=r_{\varphi^{-1}_{k,y}(h_0)^{-1}}r_{a(y,g)^{-1}}r_{a(y,g)}\varphi^{-1}_{k,y}=F(k,y)\quad\text{a.e. } k,y.$$

By ergodicity, there exists an A in $\mathcal{I}(H)$ such that $F(k,y)=A$ a.e. k,y. Hence $A\varphi_{k,y}=r_{\varphi^{-1}_{k,y}(h_0)^{-1}}$ a.e. k,y. Define $c(k,y)=\varphi^{-1}_{k,y}(h_0)$. Then $A\varphi_{k,y}=r_{c(k,y)^{-1}}$ a.e. k,y.

We then have $L_A VU(k,y) = R_{c(k,y)}$ a.e. k, y. Redefining V to be $L_A V$, we see there exists a unitary V from $L^2(K)$ to $L^2(H)$ such that $VR_k W(y) = R_{c(k,y)}$ a.e. k, y. By symmetry there exists a V' from $L^2(H)$ to $L^2(K)$ such that $V'R_h W(y)^{-1} = R_{d(h,y)}$ a.e. where $d : H \times Y \to K$ is Borel.

We claim $VR_k W(y) = R_{c(k,y)}$ for all k a.e. y. It suffices to note that if $R_{a_n} \to L_\varphi$ strongly, then $L_\varphi = R_a$ for some a. Also one has $V'R_h W(y)^{-1} = R_{d(h,y)}$ for all h a.e. y.

Hence $VV'R_h W(y)^{-1} = VR_{d(h,y)} = VR_{d(h,y)} W(y) W(y)^{-1} = R_{c(d(h,y),y)} W(y)^{-1}$ for all h a.e. y. Hence $VV'R_h = R_{c(d(h,y),y)}$ for all h a.e. y. Hence $VV' = R_{c(d(h,y),y)h^{-1}} = R_{h_0}$ a.e. y. So $c(d(h,y),y) = h_0 h$. Similarly $d(c(k,y),y) = k_0 k$ a.e. y. By redefining V to be $R_{h_0^{-1}} V$, one can see one may assume $h_0 = e$. Hence $c(d(h,y),y) = h$ for all h a.e. y. But $VR_k W(y) = R_{c(k,y)}$ and $V'R_h W(y)^{-1} = R_{d(h,y)}$ for all k and h a.e. y implies $VR_{d(h,y)} W(y) = R_h$ for all h a.e. y. Hence $V' = V^{-1}$ and $d(c(k,y),y) = k$ for all k a.e. y. Hence $c_y : K \to H$ and $d_y : H \to K$ defined by $c_y(k) = c(k,y)$ and $d_y(h) = d(h,y)$ are Borel isomorphism inverse to one another for a.e. y. Furthermore, since $VR_{kb(y,g)} W(y \cdot g) = VR_k W(y) R_{a(y,g)}$ a.e. k, y, $c(kb(y,g), y \cdot g) = c(k,y) a(y,g)$ a.e. k, y for each g.

Let m_K be a right invariant Haar measure on K. We let $\mathcal{M}(H)$ be the space of all measures on H finite on compact sets. Let $\mathcal{M}(H)$ have smallest Borel structure such that $\mu \mapsto \mu(E)$ is Borel for any Borel subset of H. Since H is second countable and locally compact, $\mathcal{M}(H)$ is a standard Borel space. Define $m(h,y) = (c_{y*} m_K) h^{-1}$ where $\mu h^{-1}(E) = \mu(Eh)$. Then $h, y \mapsto m(h,y) \in \mathcal{M}(H)$ is Borel. Furthermore,

$$\begin{aligned} m(ha(y,g), y \cdot g)(E) &= (c_{y \cdot g *} m_K)(Eha(y,g)) \\ &= m_K\{k : c(k, y \cdot g) \in Eha(y,g)\} \\ &= m_K\{k : c(kb(y,g), y \cdot g) \in Eha(y,g)\} \\ &= m_K\{k : c(k,y) \in Eh\} \\ &= m(h,y)(E) \end{aligned}$$

a.e. h, y. Hence there is an $m_H \in \mathcal{M}(H)$ such that $m(h,y) = m_H$ a.e. h, y. Hence $c_{y*} m_K h^{-1} = m_H$ a.e. h a.e. y. Therefore $c_{y*} m_K h^{-1} = m_H$ for all a.e. y. Hence $c_{y*} m_K = m_H$ a.e. y and m_H is right invariant; hence m_H is a Haar measure. Define $C : K \times Y \to H \times Y$ by $C(k,y) = (c(k,y), y)$. Then C is a Borel mapping which is essentially one-to-one, and $C_*(m_K \times \nu) = m_H \times \nu$. Furthermore

$$
\begin{aligned}
C((k,y)\cdot g) &= (c(kb(y,g),y\cdot g),y\cdot g)\\
&= (c(k,y)a(y,g),y\cdot g)\\
&= C(k,y)\cdot g \qquad \text{a.e. } k,y \quad \text{for each } g.
\end{aligned}
$$

Hence the actions are conjugate. Q.E.D.

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